

# Pricing options on illiquid assets with liquid proxies using utility indifference and dynamic-static hedging

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## Abstract

This work addresses the problem of optimal pricing and hedging of a European option on an illiquid asset  $Z$  using two proxies: a liquid asset  $S$  and a liquid European option on another liquid asset  $Y$ . We assume that the  $S$ -hedge is dynamic while the  $Y$ -hedge is static. Using the indifference pricing approach we derive a HJB equation for the value function, and solve it analytically (in quadratures) using an asymptotic expansion around the limit of the perfect correlation between assets  $Y$  and  $Z$ . While in this paper we apply our framework to an incomplete market version of the credit-equity Merton's model, the same approach can be used for other asset classes (equity, commodity, FX, etc.), e.g. for pricing and hedging options with illiquid strikes or illiquid exotic options.

## 1 Introduction

Consider a trader who wants to buy or sell a European option  $C_Z$  on asset  $Z$  with maturity  $T$  and payoff  $G_Z$ . The trader wants to hedge this position, but the underlying asset  $Z$  is illiquid. However, some liquid proxies of  $Z$  are available in the marketplace. First, there

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is a financial index (or simply an *index*)  $S$  (such as e.g. S&P500 or CDX.NA)<sup>1</sup> whose market price is correlated with  $Z$ . In addition, there is another correlated asset  $Y$  which has a liquidly traded option  $C_Y$  with a payoff  $G_Y$  similar to that of  $C_Z$ , and with the same maturity  $T$ . The market price  $p_Y$  of  $C_Y$  is also known.

Our trader realizes that hedging  $Z$ -derivative with the index  $S$  alone may not be sufficient for a number of reasons. First, she might be faced with a situation where correlation coefficients  $\rho_{yz}, \rho_{sz}$  (which for simplicity are assumed to be constant) are such that  $\rho_{yz} > \rho_{sz}$ . In this case we would intuitively expect a better hedge produced by using  $Y$  or  $C_Y$  as the hedging instruments. Second, if we bear in mind a stochastic volatility-type dynamics for  $Z$ , the stochastic volatility process may be "unspanned", i.e. the volatility risk of the option may not be traded away by hedging in option's underlying<sup>2</sup>. If that is the case, one might want to hedge the unspanned stochastic volatility by trading in a "similar" option with on the proxy asset  $Y$ . So our trader is contemplating a hedging strategy that would use both  $S$  and  $Y$ . To capture an "unspanned" stochastic volatility, the trader wants to use a derivative  $C_Y$  written on  $Y$  rather than asset  $Y$  directly.

As transaction costs are usually substantially higher for options than for underlyings, our trader sets up a static hedge in  $C_Y$  and a dynamic hedge in  $S_t$ . The static hedging strategy amounts to selling  $\alpha$  units of  $C_Y$  options at time  $t = 0$ . An *optimal* hedging strategy would be composed of a pair  $(\alpha^*, \pi_s^*)$  where  $\alpha^*$  is the optimal static hedge, and  $\pi_s^*$  (where  $0 \leq s \leq T$ ) is an optimal dynamic hedging strategy in index  $S_t$ . The pair  $(\alpha^*, \pi_s^*)$  should be obtained using a proper model. The same model should produce the highest/lowest price for which the trader should agree to buy or sell the  $Z$ -option.

In this paper we develop a model that formalizes the above scenario by supplementing it with the specific dynamics for asset prices  $S_t, Y_t$  and  $Z_t$ , and providing criteria of optimality for pricing options  $C_Z$ . For the former, we use a standard correlated log-normal dynamics. For the latter, we employ the utility indifference framework with an exponential utility, pioneered by Davis [3], Hodges & Neuberger [7] and others, see e.g. [6] for a review. As will be shown below, this results in a tractable formulations with analytical (in quadratures) expressions for optimal hedges and option prices.

As the above setting of pricing and hedging an illiquid option position using a pair of liquid proxies (e.g. a stock and an option on a different underlying) is quite general, one could visualize its potential applications for various asset classes such as equities, commodities, FX etc. For definiteness, in this paper we concentrate on a problem of practical interest for counterparty credit risk management<sup>3</sup>. Namely, we consider the

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<sup>1</sup>Here we refer to this instrument as an index, but it could be any "linear" instrument such as stock, forward, etc.

<sup>2</sup>For a discussion of such scenarios for commodities markets, see [19].

<sup>3</sup>Most of the formulae below, excluding those that use specific forms of payoffs, are general and

problem of pricing and hedging an exposure to a counterparty with an illiquid debt, and in the absence of liquidly traded CDS referencing this counterparty. For such situation, no market-implied spreads are available for the counterparty in question. Instead, one should rely on a model to come up with *theoretical* credit spreads for the counterparty. To this end, we use a version of the classical Merton equity-credit model [14] which is set up in a multi-name setting, and under the physical (i.e. "real", not "risk-neutral") measure. Most importantly, unlike the classical Merton's model, we do not intend to use firm's equity to hedge firm's debt. Instead, illiquid debt is hedged with a proxy liquid debt, and a proxy credit index. In what follows, to differentiate our framework from that of the classical Merton model, we will refer to it as the Hedged Incomplete-market Merton's Dynamics, or HIMD for short.

## 1.1 Relation to previous literature

Our model unifies three strands in the literature on indifference pricing.

The first strand deals with hedging an option with a proxy asset, as developed in Davis [4], Henderson & Hobson [5], Musiela & Zariphopoulou [15], and others. In this setting, one typically hedges an option on an illiquid underlying with a liquid proxy asset.

The second strand develops generalizations of the classical Merton credit-equity model to an incomplete market setting. Typically, this is achieved by de-correlating asset value and equity price at the level of a single firm, see e.g. Jaimungal & Sigloch [11], T. Leung & Zariphopoulou [18]. As long as we do not use firm's equity to hedge firm's debt but instead use a liquid proxy bond as a hedge, such modification of the Merton model is not needed in our setting.

The third strand is presented by [8] who develop a static-dynamic indifference hedging approach for barrier options. Ilhan and Sircar considers hedging a barrier option under stochastic volatility using static hedges in vanilla options on the same underlying plus a dynamic hedge in the underlying. This results in a two-dimensional Hamilton-Jacobi-Bellmann (HJB) equation. Our construction is similar but our hedges are a proxy asset and a proxy option, while volatility is taken constant for simplicity.

## 2 Static hedging in indifference pricing framework

Borrowing from an approach of [8] for a similar (but not identical) setting, we now show how the method of indifference utility pricing can be generalized to incorporate our scenario of a mixed dynamic-static hedge.

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applicable for other similar settings.

To this end, let  $\Pi(Y_T, Z_T)$  be the final payoff of the portfolio consisting of our option positions, i.e.

$$\Pi^\alpha(Y_T, Z_T) = G_Z - \alpha^* G_Y \quad (1)$$

As long as both European options  $C_Z, C_Y$  pay at the same maturity  $T$ , we can view this as the payoff of a combined ("static hedge portfolio") option  $g_Z^\alpha$ , which involves payoffs  $G_Z$  and  $G_Y$  of both derivatives  $C_Z$  and  $C_Y$ . Such option may be priced using the standard utility indifference principle. The latter states that the derivative price  $g_Z^\alpha$  is such that the investor should be indifferent to the choice between two investment strategies. With the first strategy, the investor adds the derivatives to her portfolio of bonds and stocks (or indices<sup>4</sup>)  $S$ , thus taking  $g_Z^\alpha$  from, and adding  $\alpha p_Y$  to her initial cash  $x$ . With the second strategy, the investor stays with the optimal portfolio containing bonds and the stocks/indices.

The value of each investment is measured in terms of the *value function* defined as the conditional expectation of utility  $U(W_T)$  of the terminal wealth  $W_T$  optimized over trading strategies. In this work, we use an exponential utility function

$$U(W) = -e^{-\gamma W} \quad (2)$$

where  $\gamma$  is a risk-aversion parameter. In our case, the terminal wealth is given by the following expression:

$$W_T = X_T + \Pi^\alpha(Y_T, Z_T)$$

with  $X_T$  be the total wealth at time  $T$  in bonds and index  $S$ . In turn, the value function reads

$$V(t, x, y, z) = \sup_{\pi_t \in \mathcal{M}} \mathbb{E} \left[ U(X_T + \Pi^\alpha(Y_T, Z_T)) \mid X_t = x, Y_t = y, Z_t = z \right] \quad (3)$$

where  $\mathcal{M}$  is a set of admissible trading strategies that require holding of initial cash  $x$ . The expectation in the Eq.(3) is taken under the "real-world" measure  $\mathbb{P}$ .

For a portfolio made exclusively of stocks/indices and bonds, the value function for the exponential utility is known from the classical Merton's work:

$$V^0(x, t) = -e^{-\gamma x e^{r\tau} - \frac{1}{2} \eta_s^2 \tau} \quad (4)$$

where  $\tau = T - t$ ,  $r$  is the risk free interest rate assumed to be constant, and  $\eta_s = (\mu_s - r)/\sigma_s$  is the stock Sharpe ratio.

In our setting, in addition to bonds and stocks/indices, we want to long  $C_Z$  option and short  $\alpha$  units of  $C_Y$  option to statically hedge our  $C_Z$  position, or, equivalently, buy the  $g_Z^\alpha$  option.

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<sup>4</sup>The stock is equivalent to our index  $S$  in the setting of the Merton's optimal investment problem.

The value function in our problem of optimal investment in bonds, index and the composite option  $g_z^\alpha$  has the following form:

$$V(x, y, z, t) = \sup_{\pi_t \in \mathcal{M}} E \left( -e^{-\gamma(X_T + \Pi^\alpha(Y_T, Z_T))} \middle| X_t = x, Y_t = y, Z_t = z \right) \quad (5)$$

where  $X_T$  is a cash equivalent of the total wealth in bonds and the index at time  $T$ . We represent it in a form similar to Eq.(4):

$$V(x, y, z, t) = -e^{-\gamma x e^{r\tau} - \frac{1}{2} \eta_s^2 \tau} \Phi(y, z, \tau) \quad (6)$$

where function  $\Phi$  will be calculated in the next sections. The indifference pricing equation reads

$$V(x, y, z, t) = V^0(x + g_Z^\alpha - \alpha p_Y, t)$$

Plugging this in Eq.(4) and Eq.(6) and re-arranging terms, we obtain

$$g_Z^\alpha = -\frac{1}{\gamma} e^{r\tau} \log \Phi(y, z, \tau) + \alpha p_Y$$

The highest price of the  $Z$ -derivative is given by choosing the optimal static hedge given by the number  $\alpha$  of the  $Y$ -derivatives, i.e.

$$\begin{aligned} g_Z^{\alpha^*} &= -\frac{1}{\gamma} e^{r\tau} \log \Phi_{\alpha^*}(y, z, \tau) + \alpha^* p_Y \\ \alpha^* &= \arg \max_{\alpha} \left\{ -\frac{1}{\gamma} e^{r\tau} \log \Phi_{\alpha}(y, z, \tau) + \alpha p_Y \right\} \end{aligned} \quad (7)$$

where we temporarily introduced subscript  $\alpha$  in  $\Phi_{\alpha}$  to emphasize that the value function depends on  $\alpha$  through a terminal condition.

### 3 The HJB equation for HIMD

To use Eq.(7) and thus be able to compute both the option price and optimal static hedge, we need to find the "reduced" value function  $\Phi$ . To this end, we first derive the Hamilton-Jacobi-Bellman (HJB) equation for our model, and then obtain its analytical (asymptotic) solution.

Let  $\pi = \pi_t(x)$  be the dynamic investment strategy in the index  $S_t$  at time  $t$  starting with the initial cash  $x$ , and  $\mathcal{L}^\pi$  be the Markov generator of price dynamics corresponding

to strategy  $\pi$ . Both the optimal dynamic strategy and the value function should be obtained as a solution of the HJB equation

$$V_t + \sup_{\pi} \mathcal{L}^{\pi} V = 0 \quad (8)$$

We assume that all state variables  $S_t, Y_t, Z_t$  follow a geometric Brownian motion process with constant drifts  $\mu_i$  and volatilities  $\sigma_i, i \in (x, y, z)$

$$\begin{aligned} dS_t &= \mu_x S_t dt + \sigma_x S_t dW_t^{(x)} \\ dY_t &= \mu_y Y_t dt + \sigma_y Y_t dW_t^{(y)} \\ dZ_t &= \mu_z Z_t dt + \sigma_z Z_t dW_t^{(z)} \end{aligned}$$

If our total wealth at time  $t$  is  $X_t = x$  and we invest amount  $\pi$  of this wealth into the index and the rest in a risk-free bond, the stochastic differential equation for  $X_t$  is obtained as follows:

$$dX_t = r(X_t - \pi) dt + \frac{\pi}{S_t} dS_t = (rX_t + \pi \sigma_x \eta_s) dt + \pi \sigma_s dW_t^{(x)}, \quad \eta_s = \frac{\mu_x - r}{\sigma_x}$$

Then  $\mathcal{L}^{\pi}$  reads

$$\begin{aligned} \mathcal{L}^{\pi} &= (rx + \pi(\mu_x - r)) V_x + \frac{1}{2} \sigma_s^2 \pi^2 V_{xx} + \mu_y y V_y + \frac{1}{2} \sigma_y^2 y^2 V_{yy} + \mu_z z V_z \\ &\quad + \frac{1}{2} \sigma_z^2 z^2 V_{zz} + \rho_{xy} \sigma_x \sigma_y \pi y V_{xy} + \rho_{xz} \sigma_x \sigma_z \pi z V_{xz} + \rho_{yz} \sigma_y \sigma_z y z V_{yz}, \end{aligned}$$

where  $V(x, y, z, t)$  is defined on the domain  $\mathbb{R}(x, y, z, t) : [0, \infty) \times [0, \infty) \times [0, \infty) \times [0, T]$ .

Since  $\mathcal{L}^{\pi}$  is a regular function of  $\pi$ ,  $\sup_{\pi}$  is achieved at

$$\pi_s^*(x) = - \frac{\eta_s V_x + \rho_{xy} \sigma_y y V_{xy} + \rho_{xz} \sigma_z z V_{xz}}{\sigma_x V_{xx}}$$

Plugging this into Eq.(8), we obtain

$$\begin{aligned} V_t + rx V_x + \mu_y y V_y + \frac{1}{2} \sigma_y^2 y^2 V_{yy} + \mu_z z V_z + \frac{1}{2} \sigma_z^2 z^2 V_{zz} + \rho_{yz} \sigma_y \sigma_z y z V_{yz} \\ - \frac{1}{2} \frac{(\eta_s V_x + \rho_{xy} \sigma_y y V_{xy} + \rho_{xz} \sigma_z z V_{xz})^2}{V_{xx}} = 0 \end{aligned} \quad (9)$$

This is a nonlinear PDE with respect to the dependent variable  $V(t, x, y, z)$  with standard boundary conditions (see [15]) and the terminal condition determined by a choice of the writer's maximal expected utility (value function) of the terminal wealth  $W_T$ .

Note that so far the derivation is valid for a generic utility function. To make further progress we specialize to the case of exponential utility in Eq.(2) since it gives rise to a natural dimension reduction of the HJB equation. Indeed, the ansatz

$$V(t, x, y, z) = -\exp(-\gamma x e^{r(T-\tau)}) G(h, s, \tau) \quad (10)$$

with  $h = \log(y/K_y)$ ,  $s = \log(z/K_z)$  is both consistent with terminal condition Eq.(5) and, upon substitution in (9), leads to a PDE for function  $G$  which does not contain variable  $x$ :

$$\begin{aligned} G_\tau = & \hat{\mu}_y G_h + \hat{\mu}_z G_s + \frac{1}{2} \sigma_y^2 G_{hh} + \frac{1}{2} \sigma_z^2 G_{ss} + \rho_{yz} \sigma_y \sigma_z G_{hs} \\ & - \frac{1}{2} \eta_s^2 G - \frac{1}{2} \frac{(\rho_{xy} \sigma_y G_h + \rho_{xz} \sigma_z G_s)^2}{G}. \end{aligned} \quad (11)$$

Here

$$\hat{\mu}_y = \mu_y - \frac{1}{2} \sigma_y^2 - \eta_s \rho_{xy} \sigma_y, \quad \hat{\mu}_z = \mu_z - \frac{1}{2} \sigma_z^2 - \eta_s \rho_{xz} \sigma_z$$

Equation Eq.(11) is defined at the domain  $\mathbb{R}(h, s, t) : [-\infty, \infty) \times [-\infty, \infty) \times [0, T]$ . The initial condition for this equation is obtained from Eq.(5).

In what follows, we choose a specific payoff of the form Eq.(1) with  $\Pi_Y = \min(Y, K_y)$ ,  $\Pi_Z = \min(Z, K_z)$  that corresponds to a portfolio of bonds of firms  $Y$  and  $Z$  with notionals  $K_y, K_z$  within the Merton credit-equity model. Then the terminal condition for  $G(h, s, \tau)$  reads

$$G(h, s, 0) = \exp[-\gamma (K_z e^{s_-} - \alpha K_y e^{h_-})] \quad (12)$$

where  $s_- = \min(s, 0)$  and  $h_- = \min(h, 0)$ .

## 4 Asymptotic solutions of Eq.(11)

We were not able to find a closed form solution of Eq.(11) with the initial condition Eq.(12). On the other hand, a numerical solution of this equation is expensive, especially when it should be used many times for calibration to market data. Therefore, we proceed with asymptotic solutions of Eq.(11). We suggest two approaches to construct asymptotic solutions.

## 4.1 First method

As we want to statically hedge option  $C_Z$  with options on another underlying, we look for an asset  $Y$  that is strongly correlated with asset  $Z$ . Further, if we have a “similar” option  $C_Y$  on asset  $Y$  (i.e. similar maturity, type, strike, etc.), we expect that such option provides a good static hedge for our option  $C_Z$  <sup>5</sup>.

Therefore, a natural assumption would be to consider  $1 - \rho_{yz}$  to be a small parameter under our setup. Utilizing this idea we represent the solution of Eq.(11) as a formal perturbative expansion in powers of  $\varepsilon$ :

$$G = \sum_{i=0}^{\infty} \varepsilon^i G_i, \quad (13)$$

where  $\varepsilon$  is the small parameter to be precisely defined in the next section.

As we shown below, Eq.(11) can be solved analytically (in quadratures) to any order of this expansion, thus significantly reducing the computation time.

### 4.1.1 The HJB equation in ”adiabatic” variables

We start with a change of variables  $(h, s) \rightarrow (w, v)$  defined as follows:

$$\begin{aligned} w &= h \frac{1}{\sigma_y} + \tau \frac{\hat{\mu}_y}{\sigma_y} \\ v &= -h \frac{1}{\sigma_y} + s \frac{\rho_{xy}}{\rho_{xz} \sigma_z} + \tau \left( -\frac{\hat{\mu}_y}{\sigma_y} + \frac{\hat{\mu}_z \rho_{xy}}{\sigma_z \rho_{xz}} \right) \end{aligned} \quad (14)$$

Simultaneously, we change the dependent variable  $G \rightarrow \Phi$  as follows:

$$G(h, s, \tau) = e^{-\frac{1}{2} \eta_s^2 \tau} \Phi(w, v, \tau) \quad (15)$$

Using Eq.(11), Eq.(14) and Eq.(15), we obtain the following PDE for function  $\Phi$ :

$$\Phi_\tau = \frac{1}{2} \Phi_{ww} + \frac{1}{2} \frac{\rho_{xz}^2 - 2\rho_{xy}\rho_{xz}\rho_{yz} + \rho_{xy}^2}{\rho_{xz}^2} \Phi_{vv} + \frac{\rho_{xz} - \rho_{xy}\rho_{yz}}{\rho_{xz}} \Phi_{wv} - \frac{1}{2} \rho_{xy}^2 \frac{\Phi_w^2}{\Phi}. \quad (16)$$

Further we will show that  $v$  is a slow (”adiabatic”) <sup>6</sup> variable of our asymptotic method, while  $w$  becomes a ”fast” variable.

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<sup>5</sup>As an example, we mention the case of equity options referencing the same underlying, i.e.  $Y = Z$ , but  $K_z \neq K_y$ . We may want to hedge an illiquid option with strike  $K_z$  (say, deep OTM) with a liquid option on the same underlying but with a different strike  $K_y$ . Under this setup, we have  $\rho_{yz} = 1$ , i.e. a perfect correlation case.

<sup>6</sup>See e.g. [? ].



In what follows we need an inverse of Eq.(14) at  $\tau = 0$ :

$$h = \sigma_y w, \quad s = (v + w) \frac{\rho_{xz} \sigma_z}{\rho_{xy}}$$

Using this in Eq.(12), we obtain the initial condition in  $(w, v)$  variables for the function  $\Phi(w, v, \tau)$ :

$$\Phi(w, v, 0) = \exp \left[ -\gamma \left( K_z e^{\frac{\rho_{xz}}{\rho_{xy}} \sigma_z (v+w)_-} - \alpha K_y e^{\sigma_y w_-} \right) \right] \quad (17)$$

where  $(x)_- = \min(x, 0)$  for any real  $x$ .

#### 4.1.2 Cosine law in 3D and "adiabatic" limit

Recall that a correlation matrix  $\Sigma$  of  $N$  assets can be represented as a Gram matrix with matrix elements  $\Sigma_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle$  where  $\mathbf{x}_i, \mathbf{x}_j$  are unit vectors on a  $N - 1$  dimensional hyper-sphere  $S^{N-1}$ . Using the 3D geometry, it is easy to establish the following *cosine law* for correlations between three assets:

$$\rho_{xy} = \rho_{yz} \rho_{xz} + \sqrt{(1 - \rho_{yz}^2)(1 - \rho_{xz}^2)} \cos(\phi_{xy}), \quad (18)$$

with  $\phi_{xy}$  being an angle between  $\mathbf{x}$  and its projection on the plane spanned by  $\mathbf{y}, \mathbf{z}$ .

As discussed e.g. by [2], three variables  $\rho_{xy}, \rho_{yz}, \phi_{xz}$  are independent, but  $\rho_{xy}, \rho_{yz}, \rho_{xz}$  are not. Therefore, one of them, e.g.  $\rho_{xy}$ , has to be found using Eq.(18) given  $\rho_{xz}, \rho_{yz}, \phi_{xy}$ .

Further we define  $\varepsilon$  as

$$\varepsilon = \sqrt{1 - \rho_{yz}^2} \ll 1, \quad (19)$$

and also define the following constants

$$\theta_1 = \frac{\sqrt{1 - \rho_{xz}^2}}{\rho_{xz}} \cos(\phi_{xy}), \quad \theta_2 = 1 + \theta_1^2 \quad (20)$$

Using Eq.(18) and definitions in Eq.(19), Eq.(20), coefficients at  $\Phi_{uv}$  and  $\Phi_{vv}$  in Eq.(16) are evaluated as follows:

$$\begin{aligned} -1 + \frac{\rho_{xy} \rho_{yz}}{\rho_{xz}} &= \varepsilon \theta_3, \quad \rho_{xy} = \rho_{xz} \beta, \quad \frac{\rho_{xz}^2 - 2\rho_{xy} \rho_{xz} \rho_{yz} + \rho_{xy}^2}{\rho_{xy}^2} = \varepsilon^2 \theta_2 \\ \beta &= \sqrt{1 - \varepsilon^2} + \varepsilon \theta_1, \quad \theta_3 = \sqrt{1 - \varepsilon^2} \theta_1 - \varepsilon. \end{aligned}$$

Accordingly using this notation Eq.(16) takes the form

$$\Phi_\tau = \frac{1}{2} \Phi_{ww} + \varepsilon \theta_3 \Phi_{wv} + \frac{1}{2} \varepsilon^2 \theta_2 \Phi_{vv} - \frac{1}{2} \rho_{xz}^2 \beta^2 \frac{\Phi_w^2}{\Phi} \quad (21)$$

In the limit  $\varepsilon \rightarrow 0$  this equation does not contain any derivatives wrt  $v$ , therefore  $v$  enters the equation only as a parameter (since  $G(w, v, \tau)$  is a function of  $v$ ). We call this limit the *adiabatic limit* in a sense that will be explained below.

It should be noted that our expansion in powers of  $\varepsilon$  can diverge if  $\rho_{xy}$  is very small. We exclude such situations on the "financial" grounds assuming that all pair-wise correlations in the triplet  $(S_t, Y_t, Z_t)$  are reasonably high (of the order of 0.4 or higher in practice), for our hedging set-up to make sense in the first place. Thus, parameter  $\theta_1$  is treated as  $O(1)$ <sup>7</sup>.

## 4.2 Second approach

It turns out that the last equation of the previous section could be further simplified. Introducing new independent variables

$$u = \frac{1}{\sqrt{\theta_2}\beta} \left( \theta_2 w - \frac{\theta_3}{\varepsilon} v \right), \quad \bar{v} = v/\varepsilon \quad (22)$$

we can transform Eq.(21) into the following equation

$$\Phi_\tau = \frac{1}{2}\Phi_{uu} + \frac{1}{2}\theta_2\Phi_{\bar{v}\bar{v}} - \frac{1}{2}\rho_{xz}^2\theta_2\frac{\Phi_u^2}{\Phi} \quad (23)$$

It is seen that in new variables the mixed derivative drops from the equation, as so does  $\varepsilon$ . However, further let us formally introduce a multiplier  $\mu$  in the term  $\Phi_{\bar{v}\bar{v}}$  which transforms Eq.(23) into

$$\Phi_\tau = \frac{1}{2}\Phi_{uu} + \frac{1}{2}\mu\theta_2\Phi_{\bar{v}\bar{v}} - \frac{1}{2}\rho_{xz}^2\theta_2\frac{\Phi_u^2}{\Phi} \quad (24)$$

Let us also formally assume that  $\mu$  is small under certain conditions. The idea of this trick is as follows. One way to construct an asymptotic solution of the Eq.(23) is to assume that all derivatives are of the order  $O(1)$ , and then estimate all the coefficients. If one manages to find a coefficient which is  $O(\mu)$ , then it is possible to build an asymptotic expansion using that coefficient (or  $\mu$  itself) as a small parameter. If, however, all the

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<sup>7</sup>While parameter  $\theta_2$  is always  $O(1)$  and positive, parameter  $\theta_1$  could be both positive and negative for typical values of correlations. For example, if  $(\rho_{xy}, \rho_{xz}, \rho_{yz}) = (0.4, 0.4, 0.8)$ , then  $\theta_1 = 0.33$ , while for  $(\rho_{xy}, \rho_{xz}, \rho_{yz}) = (0.3, 0.2, 0.8)$  it is  $\theta_1 = -0.22$ . The cosine law can also be used to find proper values of correlation parameters in the limit  $\rho_{yz} \rightarrow 1$ . To this end, we first use Eq.(18) to convert the estimated triplet  $(\rho_{xy}, \rho_{xz}, \rho_{yz})$  into a triplet of independent variables  $(\rho_{xz}, \phi_{xy}, \rho_{yz})$ , and then take the limit  $\rho_{yz} \rightarrow 1$  while keeping  $\rho_{xz}$  and  $\phi_{xy}$  constant.

coefficients in the considered PDE are of order  $O(1)$  we need to check if perhaps some of the derivatives in the Eq.(23) are small, e.g.  $O(\mu)$ . If this is the case, in order to apply standard asymptotic methods we formally have to add a small parameter  $\mu$  as a multiplier to the derivative which is  $O(\mu)$ <sup>8</sup>, make an asymptotic expansion on  $\mu$ , solve the obtained equations in every order on  $\mu$ , and at the end in the final solution put  $\mu = 1$ . That is exactly the way we want to proceed with.

This means that instead of the Eq.(13), we now have the following expansion:

$$G = \sum_{i=0}^{\infty} \mu^i G_i. \quad (25)$$

To find conditions when  $\Phi_{\bar{v}\bar{v}}$  could be small as compared with the other terms in the Eq.(23), we use an inverse map at  $\tau = 0$ :  $(h, s) \rightarrow (u, \bar{v})$ <sup>9</sup>

$$\begin{aligned} s &= \frac{\sigma_z}{\sqrt{\theta_2}} \left( \bar{v} \frac{\theta_1}{\sqrt{\theta_2}} + u \right), \\ h &= \frac{\sigma_y}{\sqrt{\theta_2}} \left( \bar{v} \frac{\theta_3}{\sqrt{\theta_2}} + u\beta \right) \end{aligned}$$

and rewrite the payoff function Eq.(12) in the form

$$\begin{aligned} \Phi(u, \bar{v}, 0) &= \exp \left[ -\gamma \left( K_z e^{\zeta \sigma_z (\omega_1 + u)-} - \alpha K_y e^{\zeta \sigma_y \beta (\omega_2 + u)-} \right) \right], \\ \omega_1 &= \bar{v} \frac{\theta_1}{\sqrt{\theta_2}}, \quad \omega_2 = \bar{v} \frac{\theta_3}{\beta \sqrt{\theta_2}}, \quad \zeta = \frac{1}{\sqrt{\theta_2}} \end{aligned} \quad (26)$$

Suppose that  $v \geq 0$ ,  $u < -\omega_1$  or  $v < 0$ ,  $u < -\omega_2$ . Differentiating the payoff twice by  $u$  and twice by  $v$  and computing the ratio of the first and second terms in the rhs of the Eq.(23), one can see that in the limit  $\varepsilon \rightarrow 0$  this ratio becomes  $\mu = \theta_2 \Phi_{\bar{v}\bar{v}} / \Phi_{uu} = \theta_1^2$ . Typical values  $\rho_{xz} = 0.3, \rho_{xy} = 0.2, \rho_{yz} = 0.8$  give rise to  $\mu = 0.05$ , therefore the second term is small as compared with the first one, and  $\mu$  is a good small parameter. This, however changes if  $\rho_{xz}$  is small or/and  $\cos(\phi_{xy})$  is close to 1, and then  $\mu \propto O(1)$ . Still in this case we have  $\varepsilon < 1$  which can be used as a small parameter. Therefore, our approach is as follows:

1. If  $\theta_1^2 \ll 1$  we use Eq.(23) and find its asymptotic solutions using Eq.(25). This is better than using Eq.(21) because first,  $\mu$  is typically smaller than  $\varepsilon$ , and second, the term  $\Phi_{\bar{v}\bar{v}}$  in our first method Eq.(13) appears only in the second order of approximation while in the second method it is taken into account already in the first order on  $\mu$ .

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<sup>8</sup>In other words write  $\Phi_{\bar{v}\bar{v}} = \mu(\Phi_{\bar{v}\bar{v}}/\mu) = \mu(\bar{\Phi}_{\bar{v}\bar{v}})$ , where  $\bar{\Phi}_{\bar{v}\bar{v}} \propto O(1)$

<sup>9</sup>At  $\varepsilon \rightarrow 0$  this is a regular map.

2. If, however,  $\theta_1^2 \propto O(1)$ , then  $\mu$  is not anymore a small parameter, therefore we use Eq.(21) and solve it asymptotically using Eq.(13).

In general, this argument cannot be applied if  $\bar{v} \geq 0$ ,  $u \geq -\omega_2$  or  $v < 0$ ,  $u \geq -\omega_1$  because then both derivatives of the payoff vanish. However, the above argument is intended to provide an intuition as to why  $\Phi_{\bar{v}\bar{v}}$  could be much smaller than the other terms in the rhs of Eq.(23). This intuition can be verified numerically, and our test examples clearly demonstrate that smallness of  $\mu$  often takes place. Below we discuss under which conditions this could occur.

Note that at the first glance, the described method looks similar to the quasi-classical approximation in quantum mechanics ([13]). The similarity comes from the observation that transformation Eq.(22) is singular in  $\varepsilon$  which is similar to the quasi-classical limit  $\hbar \rightarrow 0$ . If we would construct an asymptotic expansion on  $\varepsilon$  we would expand the rhs of the Eq.(23) on  $\varepsilon$ , but not the payoff function. After getting the solution of Eq.(23) in zero-order approximation on  $\varepsilon$  as a function of  $(u, \bar{v})$ , we would apply the inverse transformation  $(u, \bar{v}) \rightarrow (h, s)$  which is non-singular. Therefore, the final result would not contain any singularity. Since  $\mu = \theta_1^2$  is defined via  $\rho_{xz}, \rho_{xy}$  and  $\rho_{yz} = \sqrt{1 - \varepsilon^2}$ , it could seem that  $\mu = \mu(\varepsilon)$ , and we face a "quasi-classical" situation.

However, as explained above, the independent parameters are  $\rho_{yz}, \rho_{xz}$  and  $\cos(\phi_{xy})$ . Therefore, by definition  $\mu = \theta_1^2(\rho_{xz}, \cos(\phi_{xy}))$  doesn't depend on  $\rho_{yz}$ , or on  $\varepsilon$ . Thus, our two methods actually correspond to different assumptions. The first one utilizes a strong correlation between assets Z and Y. The second assumes a strong correlation between index S and asset Z while at the same time the vector of correlation  $\rho_{xz}$  in 3D space is not collinear to the vector of correlation  $\rho_{xy}$ . By financial sense this means (see Eq.(18)) the following.

1. Either  $\rho_{xz}$  is about 1 and, therefore,  $\rho_{xy} \approx \rho_{yz}$ . In other words, index S strongly correlates to asset Z, so S is almost Z, therefore correlation of Y and Z ( $\rho_{yz}$ ) is close to correlation of Y and X ( $\rho_{xy}$ ). That, in turn, means that asset Z can be dynamically hedged with S, and extra static hedge with Y doesn't bring much value. In contrast, under the former assumption static hedge plays an essential role.
2. Or  $\cos(\phi_{xy}) \ll 1$  which means that  $\rho_{xy} \approx \rho_{xz}\rho_{yz}$ . For instance,  $\rho_{xz} = 0.4$ ,  $\rho_{yz} = 0.6$ ,  $\rho_{xy} = 0.24$ . This is an interesting case, since it differs from two previously considered assumptions on high value of either  $\rho_{yz}$  or  $\rho_{xz}$ . Indeed, all correlations could be relatively moderate while providing a smallness of  $\theta_1$ .

In what follows, we describe in detail the asymptotic solutions for zero and first order approximations in  $\mu$ , and outline a generalization of our approach to an arbitrary order

in  $\mu$ . Asymptotic solutions in  $\varepsilon$  are constructed in a very similar way and are given in Appendix A. Also to make our notation lighter in the next section we will use  $v$  instead of  $\bar{v}$  since that should not bring any confusion.

### 4.3 Zero-order approximation

In the zero order approximation we set  $\mu = 0$ , so that Eq.(23) does not contain derivatives wrt  $v$ :

$$\Phi_{0,\tau} = \frac{1}{2}\Phi_{0,uu} - \frac{1}{2}\bar{\rho}_{xz}^2 \frac{(\Phi_{0,u})^2}{\Phi_0}, \quad (27)$$

where  $\bar{\rho}_{xz}^2 = \theta_2 \rho_{xz}^2$ . Therefore, dependence of the solution on  $v$  is determined by the terminal condition. In other words, our system changes along variable  $u$ , but it remains static (i.e. of the order of  $\mu^2$  slow) in variable  $v$ . Using analogy with physics, we call this limit the *adiabatic limit*.

The last equation can be solved by a change of dependent variable (closely related to the Hopf-Cole transform, see e.g. Henderson & Hobson [5], Musiela & Zariphopoulou [15]):

$$\Phi_0(\tau, u, v) = [\phi(\tau, u, v)]^{1/(1-\bar{\rho}_{xz}^2)} \quad (28)$$

which reduces Eq.(27) to the heat equation

$$\phi_\tau = \frac{1}{2}\phi_{uu}, \quad (29)$$

subject to the initial condition  $\phi_{u,v,0} = \Phi(u, v, 0)^{1-\bar{\rho}_{xz}^2}$ . The latter can be obtained from the Eq.(26) if one replaces  $\gamma$  with the "correlation-adjusted" risk aversion parameter  $\bar{\gamma} = \gamma(1 - \bar{\rho}_{xz}^2)$ . It can also be written as a piece-wise analytical function having a different form in different intervals of  $u$ -variable. If  $v \geq 0$ , we have

$$\phi(u, v, 0) = \begin{cases} \exp \left[ -\bar{\gamma} (K_z e^{\zeta \sigma_z (\omega_1 + u)} - \alpha K_y e^{\zeta \beta \sigma_y (\omega_2 + u)}) \right], & u < -\omega_1 \\ \exp \left[ -\bar{\gamma} (K_z - \alpha K_y e^{\zeta \beta \sigma_y (\omega_2 + u)}) \right], & -\omega_1 \leq u < -\omega_2 \\ \exp \left[ -\bar{\gamma} (K_z - \alpha K_y) \right], & u \geq -\omega_2, \end{cases} \quad (30)$$

while for  $v < 0$  we have

$$\phi(u, v, 0) = \begin{cases} \exp \left[ -\bar{\gamma} (K_z e^{\zeta \sigma_z (\omega_1 + u)} - \alpha K_y e^{\zeta \beta \sigma_y (\omega_2 + u)}) \right], & u < -\omega_2 \\ \exp \left[ -\bar{\gamma} (K_z e^{\zeta \sigma_z (\omega_1 + u)} - \alpha K_y) \right], & -\omega_2 \leq u < -\omega_1 \\ \exp \left[ -\bar{\gamma} (K_z - \alpha K_y) \right], & u \geq -\omega_1 \end{cases}$$

Using the well-known expression for the Green's function of our heat equation  $G_0(u' - u, \tau) = \frac{e^{-\frac{(u'-u)^2}{2\tau}}}{\sqrt{2\pi\tau}}$  (see e.g. [16]), the solution of Eq.(36) is then

$$\phi(u, v, \tau) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(u'-u)^2}{2\tau}} \phi(u', v, 0) du'$$

The explicit zero-order solution thus reads

$$\Phi_0(u, v, \tau) = \left[ \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(u'-u)^2}{2\tau}} \phi(u', v, 0) du' \right]^{1/(1-\bar{\rho}_{xz}^2)} \quad (31)$$

Note that Eq.(31) provides the general zero-order solution for the HJB equation with arbitrary initial conditions at  $\tau = 0$ . For our specific initial conditions Eq.(30), the solution is readily obtained in closed form in terms of the error (or normal cdf) function (see Appendix B). However, for numerical efficiency it might be better to use another method which is based on a simple observations that the expression in square brackets in the Eq.(31) is just a Gauss transform of the payoff. This transform can be efficiently computed using a Fast Gauss Transform algorithm which in our case is  $O(2N)$  with  $N$  being the number of grid points in  $u$  space.

#### 4.4 First-order approximation

For the first correction in Eq.(23) we obtain the following PDE

$$\Phi_{1,\tau} = \frac{1}{2}\Phi_{1,uu} - \bar{\rho}_{xz}^2 \frac{\Phi_{0,u}}{\Phi_0} \Phi_{1,u} + \frac{1}{2}\bar{\rho}_{xz}^2 \left( \frac{\Phi_{0,u}}{\Phi_0} \right)^2 \Phi_1 + \Theta_1(u, v, \tau) \quad (32)$$

with  $\Theta_1(u, v, \tau) = \theta_2 \Phi_{0,vv}/2$ . This is an inhomogeneous *linear* PDE with variable coefficients. As long as our zero-order solution of Eq.(27) already satisfies the initial condition, this equation has to be solved subject to the zero initial condition. This considerably simplifies the further construction.

We look for a solution to Eq. (32) in the form

$$\Phi_1(u, v, \tau) = [\Phi_0(u, v, \tau)]^{\bar{\rho}_{xz}^2} H(u, v, \tau)$$

This gives rise to an inhomogeneous heat equation for function  $H$  subject to zero initial condition

$$H_\tau = \frac{1}{2}H_{uu} + \Theta_1 \Phi_0^{-\bar{\rho}_{xz}^2}.$$

Thus, using the Duhamel's principle ([16]) we obtain

$$\Phi_1(u, v, \tau) = \Phi_0^{\bar{\rho}_{xz}^2}(u, v, \tau) \int_0^\tau d\chi \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \Theta_1(u', v, \chi) \Phi_0^{-\bar{\rho}_{xz}^2}(u', v, \chi) \quad (33)$$

There exists a closed form approximation of the internal integral (see Appendix D).

## 4.5 Second order approximation and higher orders

The second order equation has the same form as the Eq.(32)

$$\Phi_{2,\tau} = \frac{1}{2} \Phi_{2,uu} - \bar{\rho}_{xz}^2 \frac{\Phi_{0,u}}{\Phi_0} \Phi_{2,u} + \frac{1}{2} \bar{\rho}_{xz}^2 \left( \frac{\Phi_{0,u}}{\Phi_0} \right)^2 \Phi_2 + \Theta_2(u, v, \tau)$$

where

$$\Theta_2(u, v, \tau) = \frac{1}{2} \theta_2 \Phi_{1,vv} - \frac{1}{2} \bar{\rho}_{xz}^2 \left[ \frac{\Phi_1^2 \Phi_{0,u}^2}{\Phi_0^3} - 2 \frac{\Phi_1 \Phi_{0,u} \Phi_{1,u}}{\Phi_0^2} + \frac{\Phi_{1,u}^2}{\Phi_0} \right]$$

As this equation has to be solved also subject to zero initial conditions, the solution is obtained in the same way as above:

$$\Phi_2(u, v, \tau) = \Phi_0^{\bar{\rho}_{xz}^2}(u', v, \tau) \int_0^\tau \int_{-\infty}^\infty \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \Theta_2(u, v, \chi) \Phi_0^{-\bar{\rho}_{xz}^2}(u', v, \chi) d\chi du'$$

This shows that in higher order approximations in  $\mu$  both the type of the equation and boundary conditions stay the same. Therefore, the solution to the  $n$ -th order approximation reads

$$\Phi_n(u, v, \tau) = \Phi_0^{\bar{\rho}_{xz}^2}(u', v, \tau) \int_0^\tau \int_{-\infty}^\infty \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \Theta_n(u, v, \chi) \Phi_0^{-\bar{\rho}_{xz}^2}(u', v, \chi) d\chi du'$$

where  $\Theta_n$  can be expressed via already found solutions of order  $i, i = 1 \dots n-1$  and their derivatives on  $u$  and  $v$ . The exact representation for  $\Theta_n$  follows combinatorial rules and reads

$$\Theta_n(u, v, \tau) = \frac{1}{2} \theta_2 \Phi_{n-1,vv} - \frac{1}{2} \bar{\rho}_{xz}^2 \Xi_n,$$

where  $\Xi_n$  is a coefficient at  $\mu^{n-1}$ ,  $n > 1$  in the expansion of

$$\frac{\Phi_{0,u}^2}{\Phi_0} \left[ \frac{(1 + \beta_1(\mu))^2}{1 + \beta_2(\mu)} - \Phi_n + 2 \frac{\Phi_0}{\Phi_{0,u}} \Phi_{n,u} \right],$$

$$\beta_1 = \sum_{i=1}^{\infty} \mu^i \Phi_{i,u} / \Phi_{0,u}, \quad \beta_2 = \sum_{i=1}^{\infty} \mu^i \Phi_i / \Phi_0$$

in series on  $\mu$ . This could be easily determined using any symbolic software, e.g. Mathematica. For instance  $\Xi_3$  reads

$$\Xi_3 = \frac{(-\Phi_1\Phi_{0,u} + \Phi_0\Phi_{1,u})(\Phi_1^2\Phi_{0,u} - \Phi_0\Phi_1\Phi_{1,u} + 2\Phi_0(-\Phi_2\Phi_{0,u} + \Phi_0\Phi_{2,u}))}{\Phi_0^4}$$

The explicit representation of the solutions of an arbitrary order in quadratures is important because, per our definition of  $\mu$ , convergence of Eq.(13) is expected to be relatively slow. Indeed, if one wants the final precision to be about  $O(0.1)$  at  $\rho_{yz} = 0.8$  ( $\mu \approx 0.36$ ), the number of important terms  $m$  in expansion Eq.(25) could be rawly calculated as  $\mu^{m+1} = 0.1$ , which gives  $m = 1.25$ , while at precision 0.01 this yields  $m = 3.5$ .

Note that all integrals with  $n > 1$  do not admit a closed form representation and have to be computed numerically. Again, this could be done in an efficient manner using the Fast Gauss Transform.

## 5 Validation of the method and some examples

To verify quality of our asymptotic method we compare two sets of results. One is obtained using zero and first order approximations (being computed via a series representation and  $\Omega$  functions given in Appendix B in Eq.(41), Eq.(42) and Eq.(45), or using the Fast Gaussian Transform). Our tests showed that the number of terms in the double sum that should be kept is small, namely truncating the upper limit in  $i$  from infinity to  $i_{max} = 10$  produces nearly identical results. Therefore, the total complexity of calculation is about 45 computations of exp and Erfc functions which is very fast. A typical time required for this at a standard PC with the CPU frequency 2.3 Ghz ranges from 0.68 sec (Test 1) to 0.35 sec (Test 2) (see below).

The other test is performed using a numerical solution of Eq.(23). In doing so we use an implicit finite difference scheme built in a spirit of [12]. After the original non-linear equation is discretized to obtain the value function at the next time level, we need to solve a 2D algebraic system of equations each of which contains a non-linear term. This could be done e.g. by applying a fixed point iterative method ([17]). In other words, at the first iteration as an initial guess we plug-in into the non-linear term the solution obtained at the previous level of time. This reduces the equation to a linear one since the non-linear term is explicitly approximated at this iteration. Next we solve the resulting 2D system of equations with a block-band matrix using a 2D LU factorization. At the second iteration, the solution obtained in such a way is substituted into the non-linear term again, so again it is approximated explicitly. Then the new system of linear equations is solved and the new approximation of the solution of the original non-linear equation is obtained. We



Test	$\mu_x$	$\sigma_x$	$r$	$\rho_{yz}$	$K_z$	$\mu_z$	$\sigma_z$	$\rho_{xz}$	$z_0$	$K_y$	$\mu_y$	$\sigma_y$	$\rho_{xy}$	$y_0$
1	0.04	0.25	0.02	0.8	110	0.05	0.2	0.4	100	90	0.03	0.3	0.3	100
2	0.04	0.25	0.02	0.8	110	0.05	0.3	0.3	50	90	0.03	0.3	0.2	100

Table 1: Initial parameters used in test calculations.

continue this process until it converges. The number of iterations to needed for numerical convergence depends on gradients of the value function, which are considerably influenced by the value of  $\gamma$ . For small values of  $\gamma$  (about 0.03) we need about 1-2 iterations, while for  $\gamma \approx 0.3$ , 5-7 iterations might be necessary. For higher values of  $\gamma$ , the fixed point iteration scheme could even diverge, so another method has to be used instead. It is also important to note that we solve the non-linear equation using the dependent variable  $\log(\Phi_0)$ , rather than  $\Phi_0$  to reduce relative gradients of the solution.

Note that for *linear* 2D parabolic equations with mixed derivatives more efficient splitting schemes exist, see e.g. [9]. In principle, such schemes could be adopted to solve Eq.(23). Though Eq.(23) is a *non-linear* equation, the presence of the non-linear term does not change its type (it is still a parabolic equation), and second, does not affect stability of the scheme if we approximate it implicitly and solve the resulting non-linear algebraic equations. A more detailed description of this modification of the splitting scheme of Hout and Welfert will be presented elsewhere.

Implementation of the numerical algorithm is done similar to [10]. In typical tests we use a non-uniform finite difference grid in  $u$  and  $v$  of size 50x50 nodes, where  $-50 < u < 50$ ,  $-30 < v < 30$ . The number of steps in time depends on maturity  $T$ , because we use a fixed time step  $\delta t = 0.1$  yrs. Typical computational time for  $T = 3$  yrs on the same PC at  $\gamma = 0.03$  is 17 sec. In Fig 1 a 3D plot of the value function  $V(u, v)$  is presented for the initial parameters marked in Table 1 as 'Test 1'. We also use  $\gamma = 0.03$ ,  $\alpha = 1$ . It is seen that  $V(u, v)$  quickly goes to constant outside of a narrow region around  $u = 0$  and  $v = 0$ .

In Fig 2 the same quantities are computed for  $v_0 = 1.22$  and  $T = 10$  yrs. Here the first plot presents comparison of the numerical solution with zero and '0+1' approximations. The second plot compares the zero and first order approximation. It is seen that the first approximation makes a small correction to the zero one in the region closed to  $u = 0$ . Also both '0' and '0+1' approximations fit the exact numerical solution relatively well. This proves that our asymptotic closed form solution is robust. Results obtained with a second set of parameters (Test 2 in the Table 1) are shown in Fig 3. In Fig 4 we present price  $g_Z^\alpha$  computed in tests 1,2 as a function of  $u$ , where  $p_Y$  is Black-Scholes put price

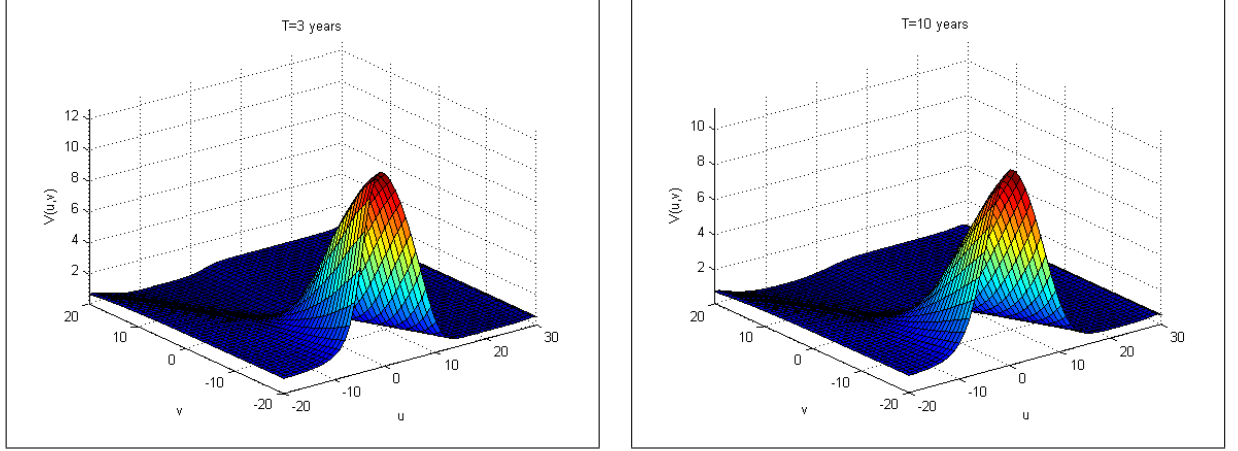


Figure 1: FD solution for the value function  $V(u, v)$  at  $T=3$  yr and  $T=10$  yrs. The initial parameters are given in Table 1, test 1.

with parameters of the corresponding tests.

Note that for values of parameters used above the nonlinear term is small, therefore the solution is closed to the solution of the linear 2D heat equation obtained from the Eq.(21) by omitting the nonlinear term. That is because  $\gamma$  is small in our Test 1. To make testing more interesting, we changed  $\gamma$  to  $\gamma = 0.2$ . Results obtained for  $T = 3$  yrs are presented in Fig 5 for Test 1. and in Fig 6 for Test 2. It is seen that in this case the '0+1' analytical approximation still fits the numerical solution.

The computational time in these tests is higher because the matrix root solver converges slower. The typical time at  $T=3$  yrs and  $\alpha = 1$  is 24 secs. Computation of the analytical approximation requires the same time as before which is about 1 sec.

If the exponent of the payoff function is positive, e.g. at  $\alpha = 2$  and other parameters as in Test 1, then the solution looks like a delta function. Under such conditions any numerical method experiences a problem being unable to resolve very high gradients within just few nodes. Therefore, in this case one has to exploit a non-uniform grid saturated close to the peak of the value function. This brings extra complexity to the numerical scheme while our analytical approximation is free of such problems.

On the other hand it is natural to statically hedge  $C_Z$  option with the option  $C_Y$  with same or close moneyness. This significantly reduces the exponent and allows one to use the same mesh even at high values of the risk aversion parameter  $\gamma$ .

As the numerical performance of the model depends on the value of  $\gamma$ , a few comments are due at this point. Though we do not address calibration of our model in the present paper (leaving it for future work), the value of  $\gamma$  should be found by calibrating the model

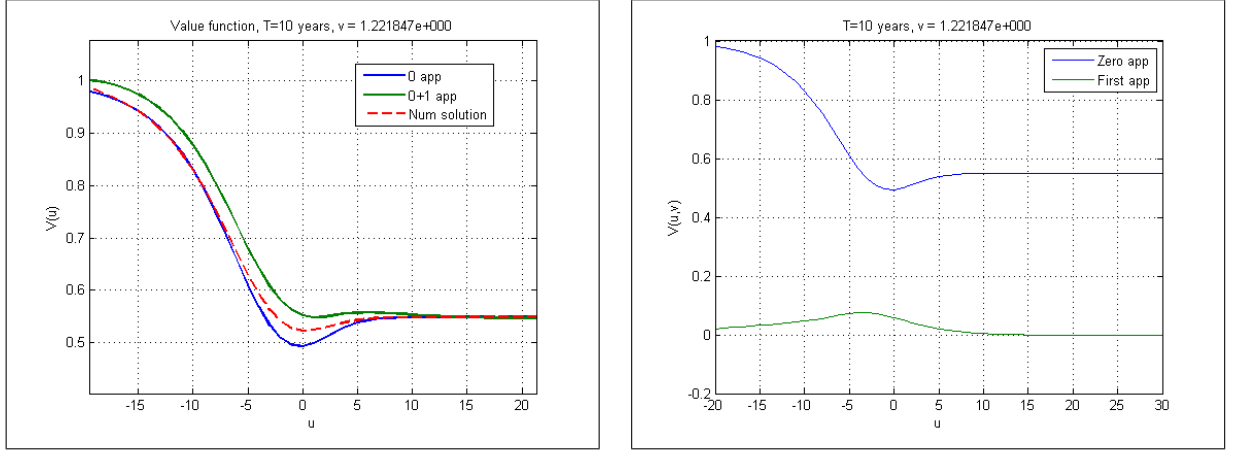


Figure 2: Value function  $V(u, v)$  at  $T=10$  yrs obtained by FD scheme, '0' and '0+1' approximations. The initial parameters are given in Table 1, test 1.

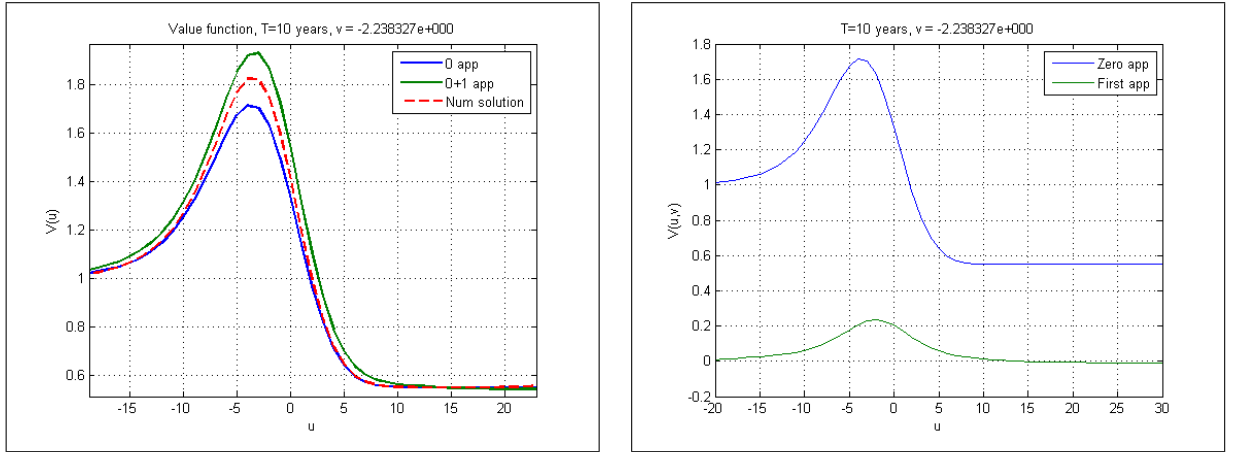


Figure 3: Value function  $V(u, v)$  at  $T=10$  yrs, comparison of the zero and first order approximations. The initial parameters are given in Table 1, test 1.

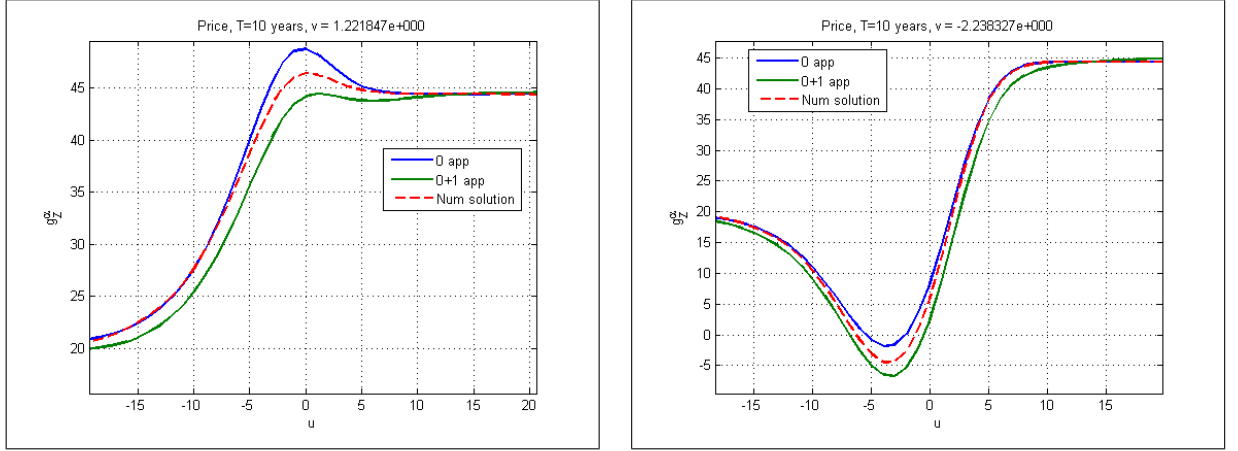


Figure 4: Price  $g_Z^\alpha$  obtained in tests 1,2.

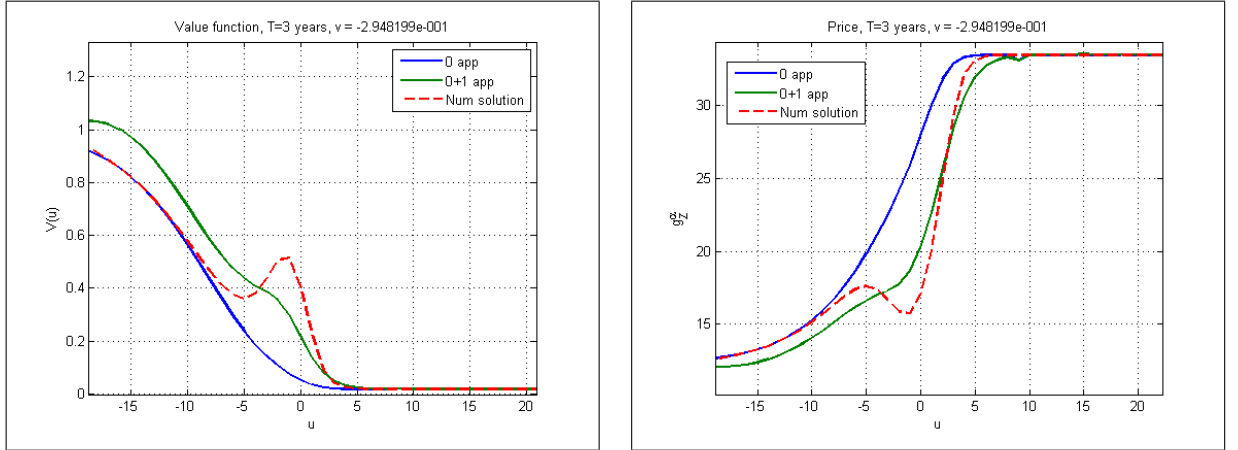


Figure 5: Value function  $V(u, v)$  and price  $g_Z^\alpha$  at  $T=3$  yrs, comparison of the numerical solution, '0' and '0+1' approximations at  $\gamma = 0.2$ . The initial parameters are given in Table 1, test 1.

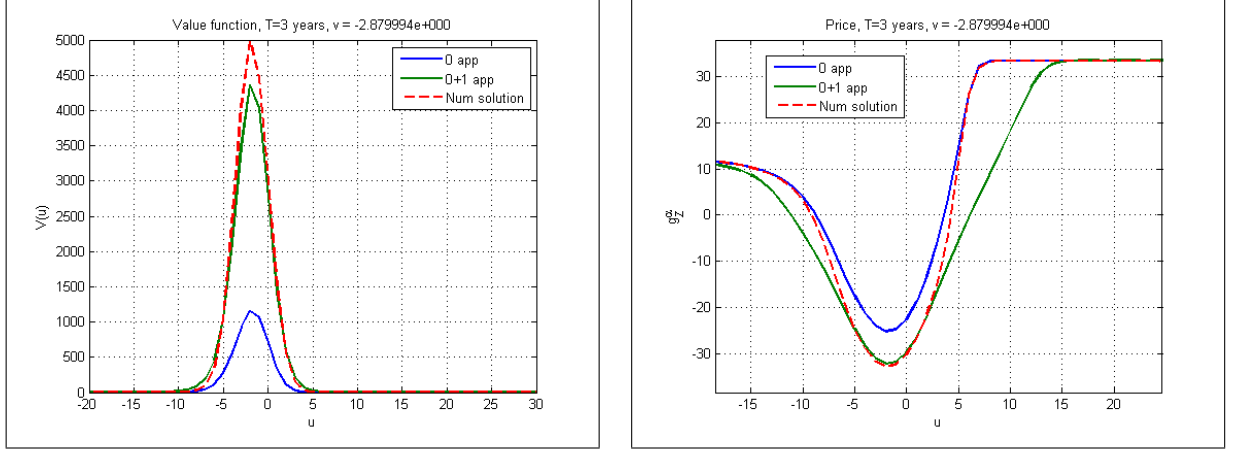


Figure 6: Value function  $V(u, v)$  and price  $g_Z^\alpha$  at  $T=3$  yrs, comparison of the numerical solution, '0' and '0+1' approximations at  $\gamma = 0.2$ . The initial parameters are given in Table 1, test 1.

to market data. It is not entirely obvious how to do this since we use an illiquid asset  $Z$ , and moreover the price of our complex option  $g_Z^\alpha$  is not an additive sum of its components for incomplete markets. It therefore makes sense to calibrate the model to another set of instruments that are both liquid and strongly correlated with the original instruments. In the context of equity options, such calibration of  $\gamma$  was done in [1], giving rise to values of  $\gamma \sim 0.1 - 0.6$ . While it is not exactly clear how  $\gamma$  found in this setting is related to  $\gamma$  of our original problem, we expect the latter to be of the same order of magnitude.

Note that the results in Fig.6 clearly demonstrate that the proposed method is just asymptotic. Indeed, the first correction to the solution in Fig.6 is a few times larger than the zero-order solution. As usual, there exists an optimal number of terms in the asymptotic series that fit the exact solution best. We do not pursue such analysis in this paper<sup>10</sup>.

To characterize sharpness of the peak in the value function it is further convenient to introduce a new parameter  $\psi = \gamma K_z$ . If  $\psi$  is high, e.g.  $\psi > 50$  the value function is almost a delta function, so the asymptotic solution as well as its numerical counterpart are not expected to produce correct results unless they are further modified. Based on the results of [1] one can see that the value of  $\psi$  calibrated to the market varies from 0.01 to 15. Therefore, in our test 2 we used  $\psi = 20$ . Both our numerical and asymptotic

<sup>10</sup>From theory we know that the asymptotic solution converges to the true solution as  $\varepsilon \rightarrow 0$ , however due to our definition of  $\varepsilon$  via  $\rho_{yz}$  this is not an interesting case.

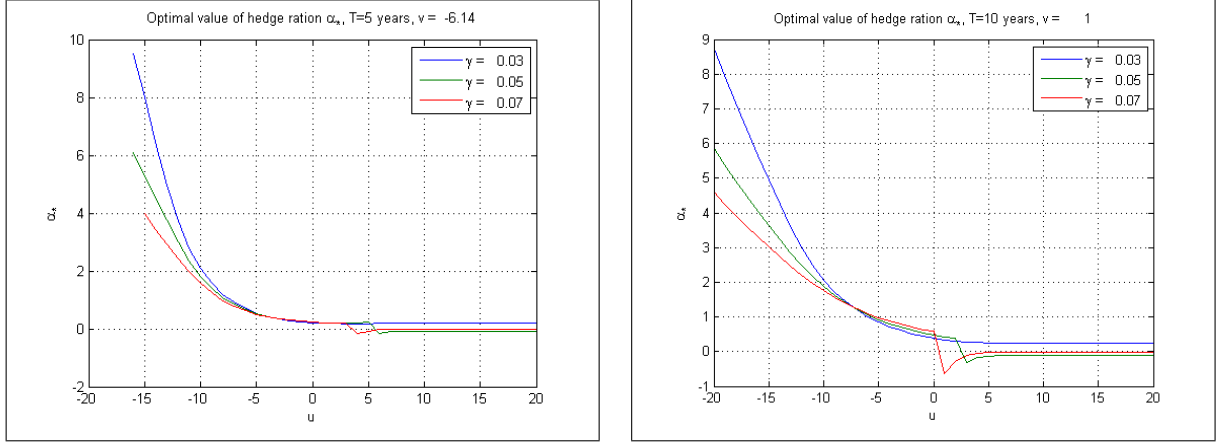


Figure 7: Optimal hedge  $\alpha_*$  computed based on Eq.(7) using the Brent method.

methods work with no problem for these values of  $\psi$ .

In Fig 7 the optimal hedge  $\alpha_*$  is computed based on Eq.(7) which was solved using Brent's method ([17]). The initial parameters correspond to test 1 in Table 1 in the first plot, and test 2 in the second plot. Note that for  $u < -15$  (the first plot) and  $u < -20$  (the second plot), Eq.(7) does not have a minimum, so the maximum is obtained at the edge of the chosen interval of  $\alpha$ . The latter could be defined based on some other preferences of the trader, for instance, the total capital she wants to invest into this strategy etc.

Finally, in Fig 8 price  $g_Z^\alpha$  is presented as a function of  $\alpha$  for various  $\gamma$ . It is seen that this function is convex which was first showed in [8] in a different setting. Note that these results were obtained using a new numerical method mentioned above. It combines Strang's splitting with the Fast Gaussian Transform, and accelerates calculations approximately by factor 40 as compared with a non-linear version of the 2d Crank-Nicholson scheme. A detailed description of the method will be given elsewhere.

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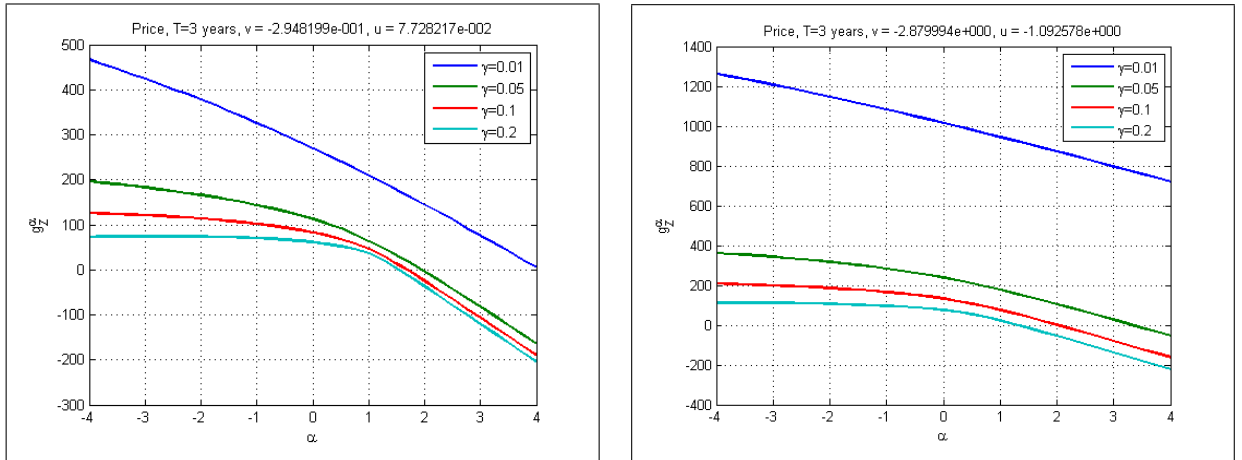


Figure 8: Price  $g_Z^\alpha$  as a function of  $\alpha$  at various  $\gamma$  computed for the initial data in tests 1 and 2.

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## A Asymptotic solutions of the Eq.(21)

Here we describe in more detail the solutions for the zero and first order approximation in  $\varepsilon$ , and outline a generalization of our approach to an arbitrary order in  $\varepsilon$ .

### A.1 Zero-order approximation

In the zero order approximation  $\varepsilon = 0$  the Eq.(21) does not contain derivatives wrt  $v$ :

$$\Phi_{0,\tau} = \frac{1}{2}\Phi_{0,uu} - \frac{1}{2}\rho_{xz}^2 \frac{(\Phi_{0,u})^2}{\Phi_0} \quad (34)$$

The solution of this equation proceeds along similar lines to Sect. 4.3 using a change of dependent variable

$$\Phi_0(\tau, u, v) = [\phi(\tau, u, v)]^{1/(1-\rho_{xz}^2)} \quad (35)$$

which reduces Eq.(34) to the heat equation

$$\phi_\tau = \frac{1}{2}\phi_{uu}, \quad (36)$$

subject to the initial condition  $\phi_{u,v,0} = \Phi(u, v, 0)^{1-\rho_{xz}^2}$ . The explicit form for the latter coincides with (30) provided we substitute  $\bar{\gamma} = \gamma(1 - \rho_{xz}^2)$ ,  $\omega_1 = 1$ ,  $\zeta = 1/\beta$  and  $\omega_2 = 0$ .

The explicit zero-order solution thus reads (compare with Eq.(31))

$$\Phi_0(u, v, \tau) = \left[ \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(u'-u)^2}{2\tau}} \phi(u', v, 0) du' \right]^{1/(1-\rho_{xz}^2)} \quad (37)$$

Note that Eq.(37) provides the general zero-order solution for the HJB equation with arbitrary initial conditions at  $\tau = 0$ . For our specific initial conditions Eq.(30), the solution is readily obtained in closed form in terms of the error (or normal cdf) function (see Appendix B).

### A.2 First-order approximation

For the first correction in the Eq.(21) we obtain the following PDE

$$\Phi_{1,\tau} = \frac{1}{2}\Phi_{1,uu} - \rho_{xz}^2 \frac{\Phi_{0,u}}{\Phi_0} \Phi_{1,u} + \frac{1}{2}\rho_{xz}^2 \left( \frac{\Phi_{0,u}}{\Phi_0} \right)^2 \Phi_1 + \Theta_1(u, v, \tau) \quad (38)$$

where  $\Theta_1(u, v, \tau) = \theta_3 \Phi_{0,uv}$ .

This equation coincides with Eq.(32) except that the free term is different, and the correlation parameter is  $\rho_{xz}$  rather than  $\bar{\rho}_{xz}$ . The solution proceeds as in Sect. 4.4, resulting in the following expression:

$$\Phi_1(u, v, \tau) = \Phi_0^{\rho_{xz}^2}(u, v, \tau) \int_0^\tau d\chi \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \Theta_1(u', v, \chi) \Phi_0^{-\rho_{xz}^2}(u', v, \chi) \quad (39)$$

The double integral that enters this expression can be split out in two parts. One of them could be found in closed form, while the other one requires numerical computation (see Appendix C). Our numerical tests show that for many sets of the initial parameters the first integral is much higher than the second one, so the latter could be neglected. However, we were not able to identify in advance at which particular values of the parameters this could be done. Moreover, for some other initial parameters we observe an opposite situation.

### A.3 Second order approximation and higher orders

The second order equation has the same form as Eq.(38)

$$\Phi_{2,\tau} = \frac{1}{2}\Phi_{2,uu} - \rho_{xz}^2 \frac{\Phi_{0,u}}{\Phi_0} \Phi_{2,u} + \frac{1}{2}\rho_{xz}^2 \left( \frac{\Phi_{0,u}}{\Phi_0} \right)^2 \Phi_2 + \Theta_2(u, v, \tau)$$

where

$$\Theta_2(u, v, \tau) = \frac{1}{2}\theta_2\Phi_{0,vv} + \theta_3(\Phi_{1,uv} - \theta_3\Phi_{0,uv}) - \frac{1}{2}\rho_{xz}^2\Phi_0(\Phi_1/\Phi_0)'_u$$

As this equation has to be solved also subject to zero initial conditions, the solution is obtained in the same way as above:

$$\Phi_2(u, v, \tau) = \Phi_0^{\rho_{xz}^2}(u', v, \tau) \int_0^\tau \int_{-\infty}^\infty \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \Theta_2(u, v, \chi) \Phi_0^{-\rho_{xz}^2}(u', v, \chi) d\chi du'$$

This shows that in higher order approximations in  $\varepsilon$  the type of the equation to solve doesn't change as well as the initial conditions. Therefore, the solution to the  $n$ -th order approximation reads

$$\Phi_n(u, v, \tau) = \Phi_0^{\rho_{xz}^2}(u', v, \tau) \int_0^\tau \int_{-\infty}^\infty \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \Theta_n(u, v, \chi) \Phi_0^{-\rho_{xz}^2}(u', v, \chi) d\chi du'$$

where  $\Theta_n$  can be expressed via already found solutions of order  $i, i = 1 \dots n - 1$  and their derivatives on  $u$  and  $v$ . The exact representation for  $\Theta_n$  follows combinatorial rules and reads

$$\begin{aligned} \Theta_n(u, v, \tau) = & \theta_3 \sum_{i=1}^n \frac{1}{(n-i+1)!} \left. \frac{\partial^{n-i+1} \xi(\varepsilon)}{\partial \varepsilon^{n-i+1}} \right|_{\varepsilon=0} \Phi_{i-1,uv} \\ & + \frac{1}{2} \theta_2 \sum_{i=2}^n \frac{1}{(n-i+2)!} \left. \frac{\partial^{n-i+2} \xi^2(\varepsilon)}{\partial \varepsilon^{n-i+2}} \right|_{\varepsilon=0} \Phi_{i-2,vv} - \frac{1}{2} \rho_{xz}^2 \Xi_n, \end{aligned} \quad (40)$$

where  $\Xi_n$  is a coefficient at  $\varepsilon^{n-1}$ ,  $n > 1$  in the following expansion

$$\Phi_0 \left( \frac{\partial \ln \Phi_0}{\partial u} \right)^2 (1 + \beta_3) \left[ 1 + \frac{\partial \ln(1 + \beta)}{\partial u} \left( \frac{\partial \ln \Phi_0}{\partial u} \right)^{-1} \right]^2, \quad \beta_3 = \sum_{i=1}^{\infty} \varepsilon^i \Phi_i / \Phi_0$$

In particular,  $\Xi_3$  reads

$$\Xi_3 = \frac{\Phi_2 \Phi_{0,u}^2 + \Phi_0 (\Phi_1^2 - 2\Phi_2 \Phi_{0,u})}{\Phi_0^2} - \frac{2\Phi_1 \Phi_{1,u}}{\Phi_{0,u}} + \frac{\Phi_0 \Phi_{1,u}^2}{\Phi_{0,u}^2} + 2\Phi_{2,u}$$

The explicit representation of the solutions of an arbitrary order in quadratures is important because per our definition of  $\varepsilon$  the convergence of the Eq.(13) is expected to be slow. Indeed, if one wants the final precision to be about 0.1 at  $\rho_{yz} = 0.8$ , the number of important terms  $m$  in the expansion Eq.(13) could be rawly calculated as  $\varepsilon^m = 0.1$ , which gives  $m = 4.5$ .

All the integrals with  $n > 1$  do not admit a closed form representation and have to be computed numerically.

## B Closed form solutions for the zero-order approximation

Since our payoff is a piece-wise function, the integral in Eq.(31) can be represented as a sum of three integrals. We denote  $\omega = \rho_{xy}v/\rho_{xz}$  and represent the zero-order solution as follows:

$$\Phi_0(u, v, \tau) = \left[ J_1^{(\zeta)} + J_2^{(\zeta)} + J_3^{(\zeta)} \right]^{\frac{1}{1-\rho_{xz}^2}}, \quad \zeta = \text{sign}(v)$$

where sign  $(-)$  means that  $\omega = \rho_{xy}v/\rho_{xz} < 0$ , and sign  $(+)$  - that  $\omega > 0$ , and

$$\begin{aligned}
J_1^{(+)} &= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{-\omega} du' e^{-\frac{(u'-u)^2}{2\tau}} \exp \left[ -\bar{\gamma} \left( K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} (\omega+u')} - \alpha K_y e^{\sigma_y u'} \right) \right] \\
&= \Omega \left( -\omega, -\bar{\gamma} K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega}, \sigma_z \frac{\rho_{xz}}{\rho_{xy}}, \bar{\gamma} \alpha K_y, \sigma_y \right), \\
J_2^{(+)} &= \frac{1}{\sqrt{2\pi\tau}} \int_{-\omega}^0 du' e^{-\frac{(u'-u)^2}{2\tau}} \exp \left[ -\bar{\gamma} \left( K_z - \alpha K_y e^{\sigma_y u'} \right) \right], \\
&= \Omega(0, -\bar{\gamma} K_z, 0, \bar{\gamma} \alpha K_y, \sigma_y) - \Omega(-\omega, -\bar{\gamma} K_z, 0, \bar{\gamma} \alpha K_y, \sigma_y), \\
J_3^{(+)} &= \frac{e^{-\bar{\gamma}(K_z - \alpha K_y)}}{\sqrt{2\pi\tau}} \int_0^{\infty} du' e^{-\frac{(u'-u)^2}{2\tau}} = e^{-\bar{\gamma}(K_z - \alpha K_y)} \left[ 1 - \frac{1}{2} \text{Erfc} \left( \frac{u}{\sqrt{2\tau}} \right) \right], \\
J_1^{(-)} &= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^0 du' e^{-\frac{(u'-u)^2}{2\tau}} \exp \left[ -\bar{\gamma} \left( K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} (\omega+u')} - \alpha K_y e^{\sigma_y u'} \right) \right] \\
&= \Omega \left( 0, -\bar{\gamma} K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega}, \sigma_z \frac{\rho_{xz}}{\rho_{xy}}, \bar{\gamma} \alpha K_y, \sigma_y \right), \\
J_2^{(-)} &= \frac{1}{\sqrt{2\pi\tau}} \int_0^{-\omega} du' e^{-\frac{(u'-u)^2}{2\tau}} \exp \left[ -\bar{\gamma} \left( K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} (\omega+u')} - \alpha K_y \right) \right] \\
&= \Omega \left( -\omega, -\bar{\gamma} K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega}, \sigma_z \frac{\rho_{xz}}{\rho_{xy}}, \bar{\gamma} \alpha K_y, 0 \right) - \Omega \left( 0, -\bar{\gamma} K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega}, \sigma_z \frac{\rho_{xz}}{\rho_{xy}}, \bar{\gamma} \alpha K_y, 0 \right), \\
J_3^{(-)} &= \frac{e^{-\bar{\gamma}(K_z - \alpha K_y)}}{\sqrt{2\pi\tau}} \int_{-\omega}^{\infty} du' e^{-\frac{(u'-u)^2}{2\tau}} = \frac{1}{2} e^{-\bar{\gamma}(K_z - \alpha K_y)} \text{Erfc} \left( -\frac{u + \omega}{\sqrt{2\tau}} \right).
\end{aligned} \tag{41}$$

Here  $\text{Erfc}(x)$  is the complementary error function, and

$$\begin{aligned}
\Omega(a, \delta, p, \beta, q) &\equiv \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^a e^{\delta e^{pu'} + \beta e^{qu'}} e^{-\frac{(u'-u)^2}{2\tau}} du' \\
&= \frac{1}{\sqrt{2\pi\tau}} \sum_{i=0}^{\infty} \sum_{j=0}^i \frac{\delta^{i-j} \beta^j}{j!(i-j)!} \int_{-\infty}^a e^{[p(i-j)+qj]u'} e^{-\frac{(u'-u)^2}{2\tau}} du' \\
&= \frac{1}{2} \sum_{i=0}^{\infty} \sum_{j=0}^i \frac{\delta^{i-j} \beta^j}{j!(i-j)!} e^{A(u+A\frac{\tau}{2})} \text{Erfc} \left( \frac{u - a + A\tau}{\sqrt{2\tau}} \right), \\
A &= ip + j(q - p).
\end{aligned} \tag{42}$$

Since the complementary error function quickly approaches zero with  $x \ll 0$ , or 1 with  $x \gg 0$ , the number of terms one has to keep in the above sums should not be high.

We also need the following function

$$\mathcal{J}_{u,v} = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(u'-u)^2}{2\tau}} \phi_{u',v}(u', v, 0) du', \quad (43)$$

when computing the first order approximation. Using a general form of the payoff  $\phi(u, v, 0) = e^{\delta(v)e^{pu} + \beta e^{qu}}$ <sup>11</sup>, it can be represented in the form

$$\begin{aligned} \mathcal{J}_{u,v} &= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(u'-u)^2}{2\tau}} \phi_{u',v}(u', v, 0) du' \\ &= \delta'(v) \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(u'-u)^2}{2\tau}} e^{pu'} \phi(u', v, 0) \left( p + e^{qu'} q \beta + e^{pu'} p \delta(v) \right) du' \\ &= \mathcal{J}_{1,u,v}^{(\zeta)} + \mathcal{J}_{2,u,v}^{(\zeta)} + \mathcal{J}_{3,u,v}^{(\zeta)}, \end{aligned} \quad (44)$$

where

$$\begin{aligned} \mathcal{J}_{2,u,v}^{(+)} &= \mathcal{J}_{3,u,v}^{(+)} = \mathcal{J}_{3,u,v}^{(-)} = 0, \\ \mathcal{J}_{1,u,v}^{(+)} &= -\bar{\gamma} K_z \sigma_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega} \Omega_1 \left( -\omega, -\bar{\gamma} K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega}, \sigma_z \frac{\rho_{xz}}{\rho_{xy}}, \bar{\gamma} \alpha K_y, \sigma_y \right) \\ \mathcal{J}_{1,u,v}^{(-)} &= -\bar{\gamma} K_z \sigma_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega} \Omega_1 \left( 0, -\bar{\gamma} K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega}, \sigma_z \frac{\rho_{xz}}{\rho_{xy}}, \bar{\gamma} \alpha K_y, \sigma_y \right), \\ \mathcal{J}_{2,u,v}^{(-)} &= -\bar{\gamma} K_z \sigma_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega} \left\{ \Omega_1 \left( -\omega, -\bar{\gamma} K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega}, \sigma_z \frac{\rho_{xz}}{\rho_{xy}}, \bar{\gamma} \alpha K_y, 0 \right) \right. \\ &\quad \left. - \Omega_1 \left( 0, -\bar{\gamma} K_z e^{\sigma_z \frac{\rho_{xz}}{\rho_{xy}} \omega}, \sigma_z \frac{\rho_{xz}}{\rho_{xy}}, \bar{\gamma} \alpha K_y, 0 \right) \right\}, \end{aligned}$$

and

$$\Omega_1(a, \delta, p, \beta, q) = \frac{1}{2} \sum_{i=0}^{\infty} \sum_{j=0}^i \frac{\delta^{i-j} \beta^j}{j!(i-j)!} [p\Lambda(p) + p\delta\Lambda(2p) + q\beta\Lambda(p+q)], \quad (45)$$

$$\Lambda(b) \equiv e^{A(b)[u+A(b)\frac{\tau}{2}]} \text{Erfc} \left( \frac{u-a+A(b)\tau}{\sqrt{2\tau}} \right), \quad A(b) \equiv ip + j(q-p) + b.$$

---

<sup>11</sup>Compare with Eq.(30).

## C Transformation of the first order solution of the Eq.(21)

The first order approximation is given by Eq.(33) where the zero-order solution  $\Phi_0(u, v, \tau)$  has been already computed in Appendix B. We plug in this solution into Eq.(33) to obtain

$$\begin{aligned}\Phi_1(u, v, \tau) &= \Phi_0^{\rho_{xz}^2}(u, v, \tau) \int_0^\tau d\chi \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \Theta_1(u', v, \chi) \Phi_0^{-\rho_{xz}^2}(u', v, \chi) \quad (46) \\ &= \theta_3 \Phi_0^{\rho_{xz}^2}(u, v, \tau) \int_0^\tau d\chi \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} J(u', v, \chi)^{\frac{-\rho_{xz}^2}{1-\rho_{xz}^2}} \partial_{u',v} J(u', v, \chi)^{\frac{1}{1-\rho_{xz}^2}}, \\ J &= \left[ J_1^{(\zeta)} + J_2^{(\zeta)} + J_3^{(\zeta)} \right]\end{aligned}$$

where the integrals  $J_i^{(\zeta)}$ ,  $i = 1, 3$  are defined in Eq.(41).

The internal integral can be simplified. Indeed, since

$$J^{\frac{-\rho_{xz}^2}{1-\rho_{xz}^2}} \partial_{u,v} J^{\frac{1}{1-\rho_{xz}^2}} = \frac{1}{1-\rho_{xz}^2} J_{u',v} + \frac{\rho_{xz}^2}{(1-\rho_{xz}^2)^2} \frac{J_v J_{u'}}{J}$$

the internal integral in Eq.(46) can be represented as a sum of two integrals  $\mathfrak{J}_1 + \mathfrak{J}_2$ , where

$$\begin{aligned}\mathfrak{J}_1 &= \frac{1}{1-\rho_{xz}^2} \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} J_{u',v}, \\ \mathfrak{J}_2 &= \frac{\rho_{xz}^2}{(1-\rho_{xz}^2)^2} \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \frac{J_v J_{u'}}{J}\end{aligned}$$

As we already mentioned the second integral could be either smaller or larger than the second one depending on the parameters. This is illustrated in Fig. 9. The initial parameters are taken from test 1 in Table 1 with  $\alpha = 1$  and  $T = 3$  yrs. In Fig. 10 the same calculation is shown for the parameters corresponding to test 2 in Table 1.

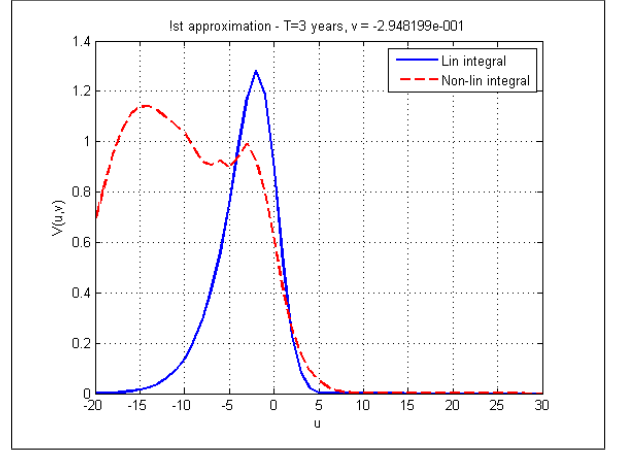
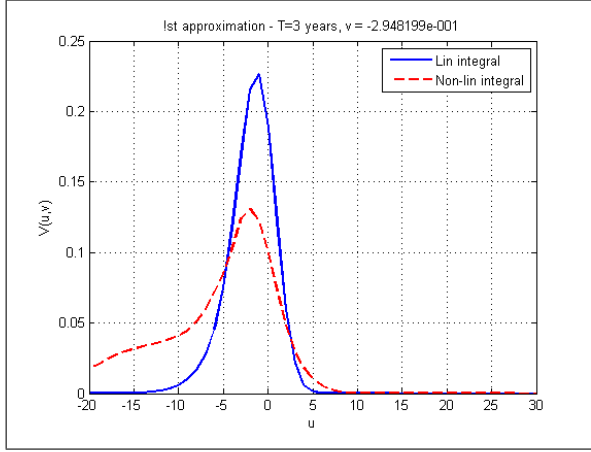


Figure 9: Comparison of  $\mathfrak{I}_1$  and  $\mathfrak{I}_2$  at  $T=3$  yr and  $\gamma = 0.03$  and  $0.2$ , test 1.

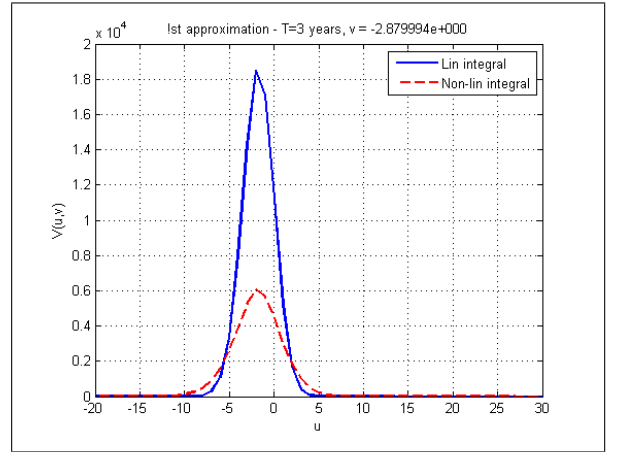
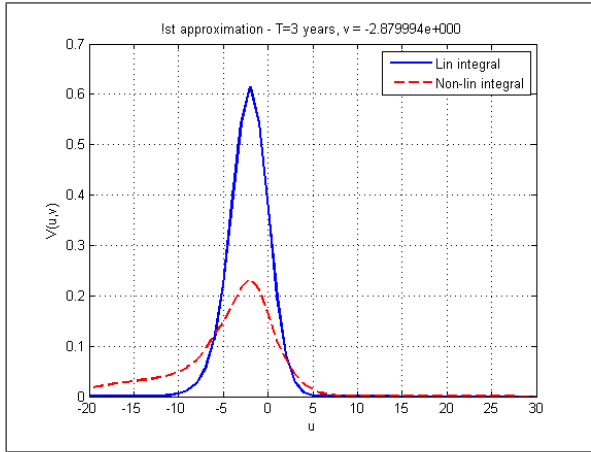


Figure 10: Comparison of  $\mathfrak{I}_1$  and  $\mathfrak{I}_2$  at  $T=3$  yr and  $\gamma = 0.03$  and  $0.2$ , test 2.

The first integral  $\mathfrak{I}_1$  can be rewritten using the integral representation of  $J$  in Eq.(31)

$$\begin{aligned}
\mathfrak{I}_1 &= \frac{1}{1 - \rho_{xz}^2} \int_{-\infty}^{\infty} du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \partial_{u'} \int_{-\infty}^{\infty} du'' \frac{e^{-\frac{(u''-u')^2}{2\chi}}}{\sqrt{2\pi\chi}} \phi_{0,v}(u'', v, 0) \\
&= \frac{1}{1 - \rho_{xz}^2} \int_{-\infty}^{\infty} du'' \phi_{0,v}(u'', v, 0) \int_{-\infty}^{\infty} du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \frac{u'' - u'}{\chi} \frac{e^{-\frac{(u''-u')^2}{2\chi}}}{\sqrt{2\pi\chi}} \\
&= \frac{1}{1 - \rho_{xz}^2} \int_{-\infty}^{\infty} du'' \phi_{0,v}(u'', v, 0) \frac{e^{-\frac{(u-u'')^2}{2\tau}}}{\sqrt{2\pi\tau^{3/2}}} (u'' - u)
\end{aligned}$$

Substituting this into Eq.(46) and integrating over  $\chi$ , we find

$$\Phi_1(u, v, \tau) = \frac{\theta_3}{1 - \rho_{xz}^2} \Phi_0^{\rho_{xz}^2}(u, v, \tau) \int_0^\tau d\chi \int_{-\infty}^{\infty} du'' \phi_{0,v}(u'', v, 0) \frac{e^{-\frac{(u-u'')^2}{2\tau}}}{\sqrt{2\pi\tau^{3/2}}} (u'' - u) \quad (47)$$

$$\begin{aligned}
&= \frac{\theta_3}{1 - \rho_{xz}^2} \Phi_0^{\rho_{xz}^2}(u, v, \tau) \int_{-\infty}^{\infty} du'' \phi_{0,v}(u'', v, 0) \frac{e^{-\frac{(u-u'')^2}{2\tau}}}{\sqrt{2\pi\tau}} (u'' - u) \\
&= -\frac{\theta_3\tau}{1 - \rho_{xz}^2} \Phi_0^{\rho_{xz}^2}(u, v, \tau) \mathcal{J}_{u,v}. \quad (48)
\end{aligned}$$

Thus, the first correction to the solution obtained in the zero order approximation on  $\varepsilon$  is approximately  $-\theta_3\tau \frac{\sqrt{1-\rho_{yz}^2}}{1-\rho_{xz}^2} \Phi_0^{\rho_{xz}^2}(u, v, \tau) \mathcal{J}_{u,v}$ , and the full solution in the "0+1" approximation reads

$$\Phi(u, v, \tau) = \Phi_0(u, v, \tau) - \theta_3\tau \frac{\sqrt{1-\rho_{yz}^2}}{1-\rho_{xz}^2} \Phi_0^{\rho_{xz}^2}(u, v, \tau) \mathcal{J}_{u,v}$$

## D Transformation of the first order solution of Eq.(23)

The first order approximation is given by Eq.(33). It is assumed that the zero-order solution  $\Phi_0(u, v, \tau)$  is already computed by using either the method presented in Appendix B,



or using the Fast Gauss Transform. We plug in this solution into Eq.(33) to obtain

$$\begin{aligned}
\Phi_1(u, v, \tau) &= \Phi_0^{\bar{\rho}_{xz}^2}(u, v, \tau) \int_0^\tau d\chi \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \Theta_1(u', v, \chi) \Phi_0^{-\bar{\rho}_{xz}^2}(u', v, \chi) \quad (49) \\
&= \frac{1}{2} \theta_2 \Phi_0^{\bar{\rho}_{xz}^2}(u, v, \tau) \int_0^\tau d\chi \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} J(u', v, \chi)^{\frac{-\bar{\rho}_{xz}^2}{1-\bar{\rho}_{xz}^2}} \partial_{v,v} J(u', v, \chi)^{\frac{1}{1-\bar{\rho}_{xz}^2}}, \\
J &= \left[ J_1^{(\zeta)} + J_2^{(\zeta)} + J_3^{(\zeta)} \right]
\end{aligned}$$

where the integrals  $J_i^{(\zeta)}$ ,  $i = 1, 3$  are defined in Eq.(41).

It can be seen that Eq.(49) is similar to Eq.(46) if one replaces  $\rho_{xz}$  with  $\bar{\rho}_{xz}$  and  $\partial_{v,v} J(u', v, \chi)^{\frac{1}{1-\bar{\rho}_{xz}^2}}$  with  $\partial_{v,v} J(v, v, \chi)^{\frac{1}{1-\bar{\rho}_{xz}^2}}$ . Therefore, we can use the same idea as in Appendix C to further simplify this integral.

Accordingly the inner integral in Eq.(49) can be rewritten as

$$J^{\frac{-\bar{\rho}_{xz}^2}{1-\bar{\rho}_{xz}^2}} \partial_{v,v} J^{\frac{1}{1-\bar{\rho}_{xz}^2}} = \frac{1}{1-\bar{\rho}_{xz}^2} J_{v,v} + \frac{\bar{\rho}_{xz}^2}{(1-\bar{\rho}_{xz}^2)^2} \frac{J_v^2}{J}$$

The internal integral in Eq.(49) can then be represented as a sum of two integrals  $\mathfrak{J}_1 + \mathfrak{J}_2$ , where

$$\begin{aligned}
\mathfrak{J}_1 &= \frac{1}{1-\bar{\rho}_{xz}^2} \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} J_{v,v}, \\
\mathfrak{J}_2 &= \frac{\bar{\rho}_{xz}^2}{(1-\bar{\rho}_{xz}^2)^2} \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \frac{J_v^2}{J}
\end{aligned}$$

The first integral  $\mathfrak{J}_1$  can be modified using the integral representation of  $J$  in Eq.(31)

$$\begin{aligned}
\mathfrak{J}_1 &= \frac{1}{1-\bar{\rho}_{xz}^2} \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \int_{-\infty}^\infty du'' \frac{e^{-\frac{(u''-u')^2}{2\chi}}}{\sqrt{2\pi\chi}} \phi_{0,vv}(u'', v, 0) \\
&= \frac{1}{1-\bar{\rho}_{xz}^2} \int_{-\infty}^\infty du'' \phi_{0,vv}(u'', v, 0) \int_{-\infty}^\infty du' \frac{e^{-\frac{(u-u')^2}{2(\tau-\chi)}}}{\sqrt{2\pi(\tau-\chi)}} \frac{e^{-\frac{(u''-u')^2}{2\chi}}}{\sqrt{2\pi\chi}} \\
&= \frac{1}{1-\bar{\rho}_{xz}^2} \int_{-\infty}^\infty du'' \phi_{0,vv}(u'', v, 0) \frac{e^{-\frac{(u-u'')^2}{2\tau}}}{\sqrt{2\pi\tau}}
\end{aligned}$$

Substituting this into Eq.(49) and integrating over  $\chi$ , we find

$$\begin{aligned}\Phi_1(u, v, \tau) &= \frac{1}{2} \frac{\theta_2}{1 - \bar{\rho}_{xz}^2} \Phi_0^{\bar{\rho}_{xz}^2}(u, v, \tau) \int_0^\tau d\chi \int_{-\infty}^\infty du'' \phi_{0,vv}(u'', v, 0) \frac{e^{-\frac{(u-u'')^2}{2\tau}}}{\sqrt{2\pi\tau}} \\ &= \frac{\tau}{2} \frac{\theta_2}{1 - \bar{\rho}_{xz}^2} \Phi_0^{\bar{\rho}_{xz}^2}(u, v, \tau) \int_{-\infty}^\infty du'' \phi_{0,vv}(u'', v, 0) \frac{e^{-\frac{(u-u'')^2}{2\tau}}}{\sqrt{2\pi\tau}}\end{aligned}\quad (50)$$

Thus, the first correction to the solution obtained in the zero order approximation on  $\mu$  is approximately  $\frac{1}{2}\tau\theta_2\frac{\mu}{1-\bar{\rho}_{xz}^2}\Phi_0^{\bar{\rho}_{xz}^2}(u, v, \tau)\mathcal{J}_{v,v}$ , and the full solution in the "0+1" approximation reads

$$\Phi(u, v, \tau) = \Phi_0(u, v, \tau) + \frac{1}{2}\tau\theta_2\frac{\mu}{1-\bar{\rho}_{xz}^2}\Phi_0^{\bar{\rho}_{xz}^2}(u, v, \tau)\mathcal{J}_{v,v}.$$